

# THE QUERMASSTEGRAL INEQUALITIES FOR $k$ -CONVEX STARSHAPED DOMAINS

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ABSTRACT. We give a proof of the isoperimetric inequality for quermassintegrals of non-convex starshaped domains, using a result of Gerhardt [6] and Urbas [13] on an expanding geometric curvature flow.

The Alexandrov-Fenchel inequalities [1, 2] for the quermassintegrals of convex domains are fundamental in classical geometry. For a bounded domain  $\Omega \subset \mathbb{R}^{n+1}$ , we denote  $M = \partial\Omega$  the boundary of  $\Omega$ . We will assume  $M$  smooth in this paper. Let

$$\kappa(x) = (\kappa_1(x), \dots, \kappa_n(x))$$

be the principal curvatures of  $x \in M$ , and let  $\sigma_k(\lambda)$  the  $k$ th elementary function in  $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$  (with  $\sigma_0(\lambda) \equiv 1$ ). There are several equivalent definitions of the quermassintegral  $V_{(n+1)-k}(\Omega)$ . For positive integer  $k$ , we will take the following

$$(1) \quad V_{(n+1)-k}(\Omega) = C_{n,k} \int_M \sigma_{k-1}(\kappa) d\mu_M,$$

where  $\sigma_k$  is the  $k$ th elementary symmetric function,

$$(2) \quad C_{n,k} = \frac{\sigma_k(I)}{\sigma_{k-1}(I)},$$

with  $I = (1, \dots, 1)$ . One may also recover  $V_{n+1}(\Omega)$  by the Minkowski type formula,

$$(3) \quad V_{(n+1)-k}(\Omega) = \int_M u \sigma_k(\kappa) d\mu_M,$$

where  $u = \langle X, \nu \rangle$ ,  $X$  is the position function of  $M$ , and  $\nu$  is the outer-normal of  $M$  at  $X$ .  $V_{n+1}(\Omega)$  is a multiple of the volume of  $\Omega$  by a dimensional constant,  $V_n(\Omega)$  is a multiple of the surface area of  $M = \partial\Omega$  by another dimensional constant. If  $\Omega$  is convex, the celebrated Alexandrov-Fenchel quermassintegral inequality states that, for  $0 \leq m \leq n$ ,

$$(4) \quad \left( \frac{V_{(n+1)-m}(\Omega)}{V_{(n+1)-m}(B)} \right)^{\frac{1}{n+1-m}} \leq \left( \frac{V_{n-m}(\Omega)}{V_{n-m}(B)} \right)^{\frac{1}{n-m}},$$

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where  $B$  is the standard ball in  $\mathbb{R}^{n+1}$ . The equality holds if and only if  $\Omega$  is a ball. The case  $m = 0$  is the classical isoperimetric inequality.

There have been some interests in extending the original Alexandrov-Fenchel inequality to non-convex domains (e.g., [12], [7]). In this short paper, we extend this inequality to starshaped domains in  $\mathbb{R}^{n+1}$ . These domains are special type of domains where fully nonlinear partial differential equations were studied in pioneer work by Caffarelli-Nirenberg-Spruck [4, 5]. We follow the notations in [12] as below.

**Definition 1.** For  $\Omega \subset \mathbb{R}^{n+1}$ , we say  $\Omega$  is  $k$ -convex if  $\kappa(x) \in \bar{\Gamma}_k$  for all  $x \in M$ , where  $\Gamma_k$  is the Garding's cone

$$\Gamma_k = \{\lambda \in \mathbb{R}^n \mid \sigma_m(\lambda) > 0, \quad \forall m \leq k\}.$$

We say  $\Omega$  is strictly  $k$ -convex if  $\kappa(x) \in \Gamma_k$  for all  $x \in M$ .

$n$ -convex is *convex* in usual sense, 1-convex is sometimes referred as *mean convex*.

**Theorem 2.** Suppose  $\Omega$  is a  $k$ -convex starshaped domain in  $\mathbb{R}^{n+1}$ , then inequality (4) is true for  $0 \leq m \leq k$ . The equality holds if and only if  $\Omega$  is a ball.

In [12], Trudinger considered inequality (4) for general the  $k$ -convex domains. He proposed an elliptic method by reducing the problem to a Hessian type equation in the domain, but the reduction argument there is incomplete. Our proof here is a parabolic one, using the flow studied by Gerhard [6] and Urbas [13]. For our purpose, we will only use a special case of their result for the following evolution equation on a hypersurface  $M_0$  in  $\mathbb{R}^{n+1}$ ,

$$(5) \quad X_t = \frac{\sigma_{k-1}}{\sigma_k}(\kappa)\nu.$$

**Theorem 3.** (Gerhardt [6], Urbas [13]) If  $\Omega_0$  is a starshaped strictly  $k$ -convex domain, then solution for flow (5) exists for all time  $t > 0$  and it converges to a round sphere after a proper rescaling.

We define

$$(6) \quad \mathcal{I}_k(\Omega) = \frac{V_{\frac{1}{(n+1)-k}}(\Omega)}{V_{\frac{1}{n-k}}(\Omega)}.$$

The key observation is that the isoperimetric ratio  $\mathcal{I}_k(\Omega)$  of the quermass-integrals are monotone along expanding flow of (5). From what follows, we will denote  $M(t)$  the solution of flow (5) at time  $t$ . If there is no confusion, we will just write  $M(t) = M$ . To simplify notation, we will also write  $\sigma_m$  for  $\sigma_m(\kappa)$  unless specified otherwise. To prepare our proof of Theorem 2, we first list the evolution equations of various geometric quantities under the following general evolution equation.

$$(7) \quad \partial_t X = F\nu,$$

where  $F = F(\kappa, X, t)$ .

**Proposition 4.** *Under flow (7), we have the following evolution equations.*

$$\begin{aligned}
 (8) \quad & \partial_t g_{ij} = 2F h_{ij} \\
 & \partial_t \nu = -\nabla F \\
 & \partial_t (d\mu_g) = F \sigma_1 d\mu_g \\
 & \partial_t h_{ij} = -\nabla_i \nabla_j F + F (h^2)_{ij} \\
 & \partial_t h_j^i = -\nabla^i \nabla_j F - F (h^2)_j^i \\
 & \partial_t \sigma_m = -\nabla_j ([T_{m-1}]_j^i \nabla_i F) - F \sigma_{m-1,1} (h_j^i; (h^2)_j^i)
 \end{aligned}$$

where we denote  $h_j^i \equiv g^{ik} h_{kj}$ ,  $(h^2)_{ij} \equiv g^{kl} h_{ik} h_{lj}$  and  $(h^2)_j^i \equiv g^{is} g^{kl} h_{sk} h_{lj}$ ,  $[T_l]_j^i$  is the  $l$ -th Newton transformation, and  $\sigma_{m-1,1}(A; B) = \frac{\partial \sigma_m}{\partial A_{ij}}(A) B_{ij}$  is a polarization of  $\sigma_m$ .

*Proof.* The proof follows from straightforward computations using the Codazzi property of the second fundamental form. The last identity follows from the divergent free property of  $[T_{m-1}]_j^i$ .  $\square$

**Lemma 5.** *Under flow (5),*

$$(9) \quad \partial_t \int_M \sigma_l d\mu_g = (l+1) \int_M \frac{\sigma_{l+1} \sigma_{k-1}}{\sigma_k} d\mu_g.$$

*Proof.* From identities in Proposition 4, for  $1 \leq l \leq n$ , we have

$$\begin{aligned}
 (10) \quad \partial_t \int_M \sigma_l d\mu_g &= \int_M \partial_t \sigma_l d\mu_g + \sigma_l \partial_t (d\mu_g) \\
 &= - \int_M \frac{\sigma_{k-1}}{\sigma_k} \left( \sigma_{l-1,1} (h_j^i; (h^2)_j^i) - \sigma_l \sigma_1 \right) d\mu_g \\
 &= (l+1) \int_M \frac{\sigma_{k-1}}{\sigma_k} \sigma_{l+1} d\mu_g,
 \end{aligned}$$

where we have used identity

$$(11) \quad \sigma_{l-1,1} (h_j^i; (h^2)_j^i) = \sigma_1 \sigma_l - (l+1) \sigma_{l+1}.$$

$\square$

There exist literatures using flow to establish geometric inequalities (e.g. [3, 8, 9]). To compare two geometric quantities, one would like to design a normalized flow so that one of the quantities is invariant under the flow, and another is monotone along the flow. Suppose we consider flow (5), and throw a time dependent constant  $R(t)$  to normalize it as the following

$$(12) \quad X_t = \left( \frac{\sigma_{k-1}}{\sigma_k}(\kappa) - R(t) \right) \nu.$$

Suppose  $\Omega_t$  is the domain with  $X(t)$  as position function. The normalization constant  $R(t)$  should be picked to make  $V_{n-k}(\Omega_t) \equiv \text{constant}$  along (12), which can be calculated easily using previous lemma. But one runs in to trouble to establish a priori estimates for flow (12) under  $k$ -convexity

assumption. By invoking the Minkowski identity, the right normalized flow should be

$$(13) \quad X_t = \left( \frac{\sigma_{k-1}}{\sigma_k}(\kappa) - r(t)u \right) \nu,$$

where  $u$  is the support function of  $M(t)$ , and

$$(14) \quad r(t) = \frac{\int_M \frac{\sigma_{k+1}\sigma_{k-1}}{\sigma_k} d\mu_g}{C_{n,k+1} \int_M \sigma_k d\mu_g}.$$

It is straightforward to show that  $r(t)$  is a normalization constant to make  $V_{n-k}(\Omega_t)$  invariant under the flow, and  $V_{n-k+1}(\Omega_t)$  is nondecreasing! Plus, one may establish all the a priori estimates for the normalized flow (13). Inequality (4) can be proved along the way.

On the other hand, it turns out that flow (13) is equivalent (up to an isomorphism) to

$$(15) \quad X_t = \frac{\sigma_{k-1}}{\sigma_k}(\kappa) \nu - r(t)X,$$

which in turn is a re-parametrization of the original flow (5). Therefore, we have the following simple proof using directly the result of Gerhardt and Urbas in Theorem 3.

**Proof of Theorem 2.** It is easy to see that  $\mathcal{I}_k(\Omega)$  is invariant under rescaling. We only need to show that,

$$(16) \quad \mathcal{I}_k(\Omega) \leq \mathcal{I}_k(B),$$

and the equality holds if and only if  $\Omega$  is a ball.

*Case 1.  $\Omega$  is strictly  $k$ -convex.*

For solution  $X(\cdot, t)$  in Theorem 3, consider  $\tilde{X}(\cdot, t) = e^{-\int_0^t r(s)ds} X(\cdot, t)$ , where  $r(t)$  as in (14). We denote  $\tilde{\Omega}_t$  to be the domain enclosed by  $\tilde{X}(\cdot, t)$ .

Since  $X(\cdot, t)$  is converging to a sphere (after a proper rescaling), we only need to show  $\mathcal{I}_k(\tilde{\Omega}_t)$  is increasing. We will continue to denote  $\sigma_m = \sigma_m(\kappa)$ , where  $\kappa$  is the principal curvature of  $X$ .

From (9) in Lemma 5, with  $C_{n,k}$  defined as in (2), we have

$$(17) \quad \begin{aligned} \frac{dV_{n-k}(\tilde{\Omega}_t)}{dt} &= (k+1)C_{n,k+1}e^{-(n-k)\int_0^t r(s)ds} \left[ \int_M \frac{\sigma_{k-1}}{\sigma_k} \sigma_{k+1} d\mu_g - rC_{n,k+1} \int_M \sigma_k d\mu_g \right] \\ &= 0, \end{aligned}$$

and

(18)

$$\begin{aligned}
\frac{dV_{(n+1)-k}(\tilde{\Omega}_t)}{dt} &= kC_{n,k}e^{-(n+1-k)\int_0^t r(s)ds} \left[ \int_M \frac{\sigma_{k-1}}{\sigma_k} d\mu_g - rC_{n,k} \int_M \sigma_{k-1} d\mu_g \right] \\
&= kC_{n,k}e^{-(n+1-k)\int_0^t r(s)ds} \int_M \left[ 1 - \frac{\int_M \frac{\sigma_{k+1}\sigma_{k-1}}{\sigma_k} d\mu_g}{C_{n,k+1} \int_M \sigma_k d\mu_g} C_{n,k} \right] \sigma_{k-1} d\mu_g \\
&\geq kC_{n,k}e^{-(n+1-k)\int_0^t r(s)ds} \int_M \left[ 1 - \frac{\sigma_{k+1}(I)\sigma_{k-1}(I)}{\sigma_k^2(I)} \frac{C_{n,k}}{C_{n,k+1}} \right] \sigma_{k-1} d\mu_g \\
&= 0,
\end{aligned}$$

where we have used the Newton-MacLaurin inequality in the last step of (18).

If the equality holds in (16), we must have  $\frac{dV_{(n+1)-k}(\tilde{\Omega}_t)}{dt} \equiv 0$ . Therefore, the equality of the Newton-MacLaurin inequality must be held at every point of  $M$  in (18). This implies  $M$  is a round sphere for each  $t \geq 0$ . In particular,  $M_0$  is a sphere.

*Case 2. General  $k$ -convex starshaped domain  $\Omega$ .*

We may approximate it by strictly  $k$ -convex starshaped domains. The inequality follows from the approximation. We now treat the equality case. We first note that both  $\int_M \sigma_k d\mu_g$  and  $\int_M \sigma_{k-1} d\mu_g$  are positive, since there exists at least one elliptic point on an embedded compact hypersurface in Euclidean space and also the  $k$ -convexity condition. Suppose  $\Omega$  is a  $k$ -convex starshaped domain with equality in (16) attained. Let  $M_+ = \{x \in M \mid \sigma_k(\kappa(x)) > 0\}$ .  $M_+$  is open and nonempty since  $M$  is compact and embedded in  $\mathbb{R}^{n+1}$ . We claim that  $M_+$  is closed. This would imply  $M = M_+$ , so  $\Omega$  is strictly  $k$ -convex, by *Case 1*, we may conclude  $\Omega$  is a standard ball.

We now prove that  $M_+$  is closed. Pick any  $\rho \in C_0^2(M_+)$  compactly supported in  $M_+$ . Let  $M_s$  be the hypersurface determined by position function  $X_s = X + s\rho\nu$ , where  $X$  is the support function of  $M$  and  $\nu$  is the unit outnormal of  $M$  at  $X$ . Let  $\Omega_s$  be the domain enclosed by  $M_s$ . It is easy to show  $M_s$  is  $k$ -convex starshaped when  $s$  is small enough. Therefore  $\mathcal{I}_k(\Omega_s) - \mathcal{I}_k(\Omega) \leq 0$  for  $s$  small, i.e.

$$\frac{d}{ds} \mathcal{I}_k(\Omega_s)|_{s=0} = 0.$$

Simple calculation yields

$$\frac{d}{ds} \int_{M_s} \sigma_l(\kappa_s) d\mu_{g_s}|_{s=0} = (l+1) \int_M \sigma_{l+1}(\kappa) \rho d\mu_g.$$

Therefore, we have

$$\frac{d}{ds} \mathcal{I}_k(\Omega_s)|_{s=0} = A \int_M (\sigma_{k+1}(\kappa) - c_1 \sigma_k(\kappa)) \rho d\mu_g = 0,$$

for some constant  $A > 0$  with  $c_1 = \frac{k(n-k)}{(k+1)(n-k+1)} \frac{1}{\mathcal{I}(B)^{n-k+1} (\int_M \sigma_k)^{\frac{1}{n-k}}} > 0$  and

for all  $\rho \in C_0^2(M_+)$ .

In turn,

$$(19) \quad \sigma_{k+1}(\kappa(x)) = c_1 \sigma_k(\kappa(x)), \quad \forall x \in M_+.$$

By the Newton-MacLaurine inequality, there is a dimensional constant  $\tilde{C}_{k,n}$  such that  $\sigma_{k+1}(\kappa(x)) \leq \tilde{C}_{k,n} \sigma_k^{1+1/k}(\kappa(x))$  for all  $x \in M_+$ . In view of (19), there is a positive constant  $c_2$ , such that

$$(20) \quad \sigma_k(\kappa(x)) \geq c_2 > 0, \quad \forall x \in M_+,$$

where  $c_2 = (\frac{c_1}{\tilde{C}_{k,n}})^k$  is a positive constant depending only on  $n$ ,  $k$ , and  $\Omega$ . (20) implies  $M_+$  is closed.  $\square$

The question of validity of inequality (4) for general  $k$ -convex domains is still open. When  $k = 1$ , flow (5) is exactly the inverse mean curvature flow. There is a notion of weak solution studied by Huisken-Ilmanen for the Penrose inequality in [11]. Huisken [10] has proved the following with outward minimising assumption.

**Theorem 6.** *If  $\Omega \subset R^{n+1}$  is outward minimising for  $n \leq 6$ , then inequality (4) is valid for  $m = 1$ .*

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